Parametric Duality and Kernelization: Lower Bounds and Upper Bounds on Kernel Size

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Abstract. We develop new techniques to derive lower bounds on the kernel size for certain parameterized problems. For example, we show that unless $\mathcal{P} = \mathcal{NP}$, PLANAR VERTEX COVER does not have a problem kernel of size smaller than $4k/3$, and PLANAR INDEPENDENT SET and PLANAR DOMINATING SET do not have kernels of size smaller than $2k$. We derive an upper bound of $67k$ on the problem kernel for PLANAR DOMINATING SET improving the previous $335k$ upper bound by Alber et al..

Keywords. kernelization, parameterized complexity

1 Introduction

Many problems which are parameterized tractable become intractable when the parameter is "turned around" (see [9, 12, 14]). As an example, consider the VERTEX COVER and INDEPENDENT SET problems. If n denotes the number of vertices in the whole graph G, then it is well-known that (G, k) is a YES-instance of VERTEX COVER if and only if (G, k_d), where $k_d = n - k$, is a YES-instance of INDEPENDENT SET. In this sense, INDEPENDENT SET is the parametric dual problem of VERTEX COVER. While VERTEX COVER is fixed-parameter tractable on general graphs, INDEPENDENT SET is not. Similarly, while DOMINATING SET is fixed-parameter intractable on general graphs, its parametric dual, called NONBLOCKER, is fixed-parameter tractable.

The landscape changes when we turn our attention towards special graph classes, e.g., problems on *planar graphs* [2]. Here, for example, both INDEPENDENT SET and DOMINATING SET are fixed-parameter tractable. In fact, and in

[†] This research was supported in part by the NSF under grants CCF-0430683 and CCR-0311590.

[‡] This work was supported in part by DePaul University Competitive Research Grant.

contrast to what was stated above, there are quite many problems for which both the problem itself and its dual are parameterized tractable.

The beauty of problems which are together with their dual problems fixed-parameter tractable, is that this constellation allows from an algorithmic standpoint for a two-sided attack on the original problem. This two-sided attack enabled us to derive lower bounds on the kernel size for such problems (under classical complexity assumptions). For example, we show that unless $\mathcal{P} = \mathcal{NP}$, PLANAR VERTEX COVER does not have a kernel of size smaller than $4k/3$, and PLANAR INDEPENDENT SET and PLANAR DOMINATING SET do not have kernels of size smaller than $2k$. To the authors' knowledge, this is the first group of results establishing lower bounds on the kernel size of parameterized problems.

Whereas the lower bounds on the kernel size for PLANAR VERTEX COVER and PLANAR INDEPENDENT SET come close to the known upper bounds of $2k$ and $4k$ on the kernel size for the two problems, respectively, the lower bound derived for PLANAR DOMINATING SET is still very far from the $335k$ upper bound on the problem kernel (computable in $O(n^3)$ time), which was given by Alber et al. [1]. To bridge this gap, we derive better upper bounds on the problem kernel for PLANAR DOMINATING SET. We improve the reduction rules proposed in [1], and introduce new rules that *color* the vertices of the graph enabling us to observe many new combinatorial properties of its vertices. These properties allow us to prove a much stronger bound on the number of vertices in the reduced graph. We show that the PLANAR DOMINATING SET problem has a kernel of size $67k$ that is computable in $O(n^3)$ time. This is a significant improvement over the results in [1].

2 Preliminaries

A *parameterized problem* P is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a fixed alphabet and $\mathbb{N}$ is the set of all non-negative integers. Therefore, each instance of the parameterized problem P is a pair (I, k), where the second component k is called the *parameter*. The language $L(P)$ is the set of all YES-instances of P. We say that the parameterized problem P is *fixed-parameter tractable* [7] if there is an algorithm that decides whether an input (I, k) is a member of $L(P)$ in time $f(k)|I|^c$, where c is a fixed constant and $f(k)$ is a recursive function independent of the input length $|I|$. The class of all fixed parameter tractable problems is denoted by FPT.

A mapping $s : \Sigma^* \times \mathbb{N} \to \mathbb{N}$ is called a *size function* for a parameterized problem P if:

- $0 \leq k \leq s(I, k)$,
- $s(I, k) \leq |I|$, and
- $s(I, k) = s(I, k')$ for all appropriate k, k' *(independence)*. Hence, we can also write $s(I)$ for $s(I, k)$.

A problem P together with its size function s are denoted (P, s). The *dual problem* P_d of P is the problem whose corresponding language (i.e., set of YES-instances) $L(P_d) = \{(I, s(I) - k) \mid (I, k) \in L(P)\}$. The dual of the dual of a problem (with a given size function) is again the original problem. We give some examples below.

d-HITTING SET
Given: A hypergraph $G = (V, E)$ with *edge degree* bounded by d, i.e., $\forall e \in E, |e| \leq d$
Parameter: k
Question: Is there a *hitting set* of size at most k, i.e.,

$$\exists C \subseteq V, |C| \leq k, \forall e \in E, C \cap e \neq \emptyset?$$

The special case in which $d = 2$ corresponds to the VERTEX COVER problem in undirected graphs. Let $L(d\text{-HS})$ denote the language of d-HITTING SET. Taking as size function $s(G) = |V|$, it is clear that the dual problem obeys $(G, k_d) \in L(d\text{-HS}_d)$ if and only if G has an *independent set* of cardinality k_d.

DOMINATING SET
Given: A (simple) graph $G = (V, E)$
Parameter: k
Question: Is there a *dominating set* of size at most k, i.e.,

$$\exists D \subseteq V, |D| \leq k, \ \forall v \in V \setminus D \ \exists d \in D, (d, v) \in E?$$

Taking as size function $s(G) = |V|$, it is clear that the dual problem obeys $(G, k_d) \in L(\text{DS}_d)$ if and only if G has a *nonblocker set* (i.e., the complement of a dominating set) of cardinality k_d.

Generally speaking, it is easy to "correctly" define the dual of a problem for selection problems as formalized in [3].

A *kernelization* for a parameterized problem P with size function s is a polynomial-time computable reduction which maps an instance (I, k) onto (I', k') such that: (1) $s(I') \leq g(k)$ (g is a recursive function), (2) $k' \leq k$, and (3) $(I, k) \in L(P)$ if and only if $(I', k') \in L(P)$. I' is called the *problem kernel* of I. It is known (see [8]) that a parameterized problem is fixed-parameter tractable if and only if it has a kernelization. Of special interest to us in this paper are problems with *linear kernels* in which $g(k) = \alpha k$ for some constant $\alpha > 0$. Such small kernels are known, in particular, for graph problems restricted to planar graphs.

3 Lower bounds on kernel size

Practice in the study of parameterized algorithms has suggested that improved kernelization can lead to improved parameterized algorithms. Many efforts have been made towards obtaining smaller kernels for well-known $\mathcal{NP}$-hard parameterized problems (see for example [1, 5, 8]). A natural question to ask along this line of research, is about the limit of polynomial time kernelization. In this section we develop techniques for deriving lower bounds on the kernel size for certain well-known $\mathcal{NP}$-hard parameterized problems.

Theorem 1. *Let (P, s) be an $\mathcal{NP}$-hard parameterized problem. Suppose that P admits an αk kernelization, and its dual P_d admits an $\alpha_d k_d$ kernelization, where $\alpha, \alpha_d \geq 1$. If $(\alpha - 1)(\alpha_d - 1) < 1$ then $\mathcal{P} = \mathcal{NP}$.*

Proof. Suppose that the statement of the theorem is true, and let $r(\cdot)$ denote the assumed linear kernelization reduction for P. Similarly, let $r_d(\cdot)$ be the linear kernelization reduction for P_d. Consider the following reduction R, which on input (I, k) of P performs the following:

if $k \leq \frac{\alpha_d}{\alpha+\alpha_d} s(I)$ **then** compute $r(I, k)$;
else compute $r_d(I, s(I) - k)$.

Now if $k \leq \frac{\alpha_d}{\alpha+\alpha_d} s(I)$, then $s(I') \leq \alpha k \leq \frac{\alpha\alpha_d}{\alpha+\alpha_d} s(I)$. Otherwise:

$$\begin{aligned} s(I') &\leq \alpha_d k_d \\ &= \alpha_d (s(I) - k) \\ &< \alpha_d \left(s(I) - \frac{\alpha_d}{\alpha + \alpha_d} s(I) \right) \\ &= \frac{\alpha\alpha_d}{\alpha + \alpha_d} s(I). \end{aligned}$$

Since $(\alpha - 1)(\alpha_d - 1) < 1$, or equivalently $\frac{\alpha\alpha_d}{\alpha+\alpha_d} < 1$, by repeatedly applying R (at most polynomially-many times), the problem P can be solved in polynomial time. This completes the proof.

From the previous theorem, and assuming $\mathcal{P} \neq \mathcal{NP}$, we immediately obtain the following.

1. **Corollary 1.** For any $\epsilon > 0$, there is no $(4/3 - \epsilon)k$ kernel for PLANAR VERTEX COVER.

   Proof. The four-color theorem implies a $4k$-kernelization for PLANAR INDEPENDENT SET, which is the dual problem of PLANAR VERTEX COVER.

2. **Corollary 2.** For any $\epsilon > 0$, there is no $(2 - \epsilon)k$ kernel for PLANAR INDEPENDENT SET. This result remains true if we restrict the problem to graphs of maximum degree bounded by three, or even to planar graphs of maximum degree bounded by three (both problems are $\mathcal{NP}$-hard).

   Proof. The general VERTEX COVER problem, which is the dual of the INDEPENDENT SET problem, has a $2k$-kernelization [5]. This kernelization is both planarity and bounded-degree preserving.

3. **Corollary 3.** For any $\epsilon > 0$, there is no $(3/2 - \epsilon)k$-kernelization for VERTEX COVER restricted to triangle-free planar graphs (this problem is still $\mathcal{NP}$-hard [15, Chapter 7]).

   Proof. Based on a theorem by Grötzsch (which can be turned into a polynomial-time coloring algorithm; see [11]) it is known that planar triangle-free graphs are 3-colorable. This implies a $3k$ kernel for INDEPENDENT SET restricted to this graph class, which gives the result. Observe that the $2k$-kernelization for VERTEX COVER on general graphs preserves planarity and triangle-freeness, which implies that this restriction of the problem has a $2k$-kernelization.

4. **Corollary 4.** For any $\epsilon > 0$, there is no $(335/334 - \epsilon)k$ kernel for PLANAR NONBLOCKER.

   Proof. A $335k$ kernel for PLANAR DOMINATING SET was derived in [1].

5. **Corollary 5.** For any $\epsilon > 0$, there is no $(2 - \epsilon)k$ kernel for PLANAR DOMINATING SET. This remains true when further restricting the graph class to planar graphs of maximum degree three (the problem is still $\mathcal{NP}$-hard).

   Proof. In [10], a $2k$-kernelization for NONBLOCKER on general graphs which preserves planarity and degree bounds, was derived (see also [13, Theorem 13.1.3]).

The above results open a new line of research, and prompt us to ask whether we can find examples of problems such that the derived kernel sizes are optimal (unless $\mathcal{P} = \mathcal{NP}$), and whether we can close the gaps between the upper bounds and lower bounds on the kernel size more and more. According to our previous discussion, PLANAR VERTEX COVER on triangle-free graphs is our "best match:" we know how to derive a kernel of size $2k$, and (assuming $\mathcal{P} \neq \mathcal{NP}$) we know that no kernel smaller than $3k/2$ exists. On the other hand, the $335k$ upper bound on the kernel size for PLANAR DOMINATING SET [1] is very far from the $2k$ lower bound proved above. In the next section, we improve this upper bound to $67k$ in an effort to bridge the huge gap between the upper bound and lower bound on the kernel size for this problem.

4 Reduction and coloring rules for PLANAR DOMINATING SET

In this section we will only consider planar graphs. For a graph G, we denote by $\gamma(G)$ the size of a minimum dominating set in G. We present an $O(n^3)$ time preprocessing scheme that reduces the graph G to a graph G', such that $\gamma(G) = \gamma(G')$, and such that given a minimum dominating set for G', a minimum dominating set for G can be constructed in linear time. We will color the vertices of the graph G with two colors: black and white. Initially, all vertices are colored black. Informally speaking, white vertices will be those vertices that we know for sure when we color them that there exists a minimum dominating set for the graph excluding all of them. The black vertices are all other vertices. Note that it is possible for white vertices to be in some minimum dominating set, but the point is that there exists at least one minimum dominating set that excludes all white vertices. We start with the following definitions that are adopted from [1] with minor additions and modifications.

For a vertex v in G denote by $N(v)$ the set of neighbors of v, and by $N[v]$ the set $N(v) \cup \{v\}$. By removing a vertex v from G, we mean removing v and all the edges incident on v from G. For a vertex v in G, we partition its set of neighbors $N(v)$ into three sets: $N_1(v) = \{u \in N(v) \mid N(u) - N[v] \neq \emptyset\}$; $N_2(v) = \{u \in N(v) - N_1(v) \mid N(u) \cap N_1(v) \neq \emptyset\}$; and $N_3(v) = N(v) - (N_1(v) \cup N_2(v))$. For two vertices v and w we define $N(v, w) = N(v) \cup N(w)$ and $N[v, w] = N[v] \cup N[w]$. We partition $N(v, w)$ into three sets: $N_1(v, w) = \{u \in N(v, w) \mid N(u) - N[v, w] \neq \emptyset\}$; $N_2(v, w) = \{u \in N(v, w) - N_1(v, w) \mid N(u) \cap N_1(v, w) \neq \emptyset\}$; and $N_3(v, w) = N(v, w) - (N_1(v, w) \cup N_2(v, w))$.

Definition 1. Let $G = (V, E)$ be a plane graph. A *region* $R(v, w)$ between two vertices v and w is a closed subset of the plane with the following properties:

1. The boundary of $R(v, w)$ is formed by two simple paths P_1 and P_2 in V which connect v and w, and the length of each path is at most three.
2. All vertices that are strictly inside (i.e., not on the boundary) the region $R(v, w)$ are from $N(v, w)$.

For a region $R = R(v, w)$, let $V[R]$ denote the vertices in R, i.e.,

$$V[R] := \{u \in V \mid u \text{ sits inside or on the boundary of } R\}.$$

Let $V(R) = V[R] - \{v, w\}$.

Definition 2. A region $R = R(v, w)$ between two vertices v and w is called *simple* if all vertices in $V(R)$ are common neighbors of both v and w, that is, $V(R) \subseteq N(v) \cap N(w)$.

We introduce the following definitions.

Definition 3. A region $R = R(v, w)$ between two vertices v and w is called *quasi–simple* if $V[R] = V[R'] \cup R^+$, where $R' = R'(v, w)$ is a simple region between v and w, and R^+ is a set of white vertices satisfying the following conditions:

1. Every vertex of R^+ sits strictly inside R'.
2. Every vertex of R^+ is connected to v and not connected to w, and is also connected to at least one vertex on the boundary of R' other than v.

A vertex in $V(R)$ is called a *simple* vertex, if it is connected to both v and w, otherwise it is called *non–simple*. The set of vertices R^+, which consists of the non-simple vertices in $V(R)$, will be referred to as $R^+(v, w)$.

For a vertex $u \in V$, denote by $B(u)$ the set of black vertices in $N(u)$, and by $W(u)$ the set of white vertices in $N(u)$. We describe next the reduction and coloring rules to be applied to the graph G. The reduction and coloring rules are applied to the graph in the order listed below until the application of any of them does not change the structure of the graph nor the color of any vertex in the graph. The first two reduction rules, **Rule 1** and **Rule 2**, are slight modifications of Rule 1 and Rule 2 introduced in [1]. The only difference is that in the current paper they are only applied to black vertices, and not to all the vertices as in [1].

Rule 1 ([1]). If $N_3(v) \neq \emptyset$ for some black vertex v, then remove the vertices in $N_2(v) \cup N_3(v)$ from G, and add a new white vertex v' and an edge (v, v') to G.

Rule 2 ([1]). If $N_3(v, w) \neq \emptyset$ for two black vertices v, w, and if $N_3(v, w)$ cannot be dominated by a single vertex in $N_2(v, w) \cup N_3(v, w)$, then we distinguish the following two cases.

Case 1. If $N_3(v, w)$ can be dominated by a single vertex in $\{v, w\}$ then: (1.1) if $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \subseteq N(w)$, remove $N_3(v, w)$ and $N_2(v, w) \cap N(v) \cap N(w)$ from G and add two new white vertices z, z' and the edges $(v, z), (w, z), (v, z'), (w, z')$ to G; (1.2) if $N_3(v, w) \subseteq N(v)$ and $N_3(v, w) \not\subseteq N(w)$, remove $N_3(v, w)$ and $N_2(v, w) \cap N(v)$ from G and add a new white vertex v' and the edge (v, v') to G; and (1.3) if $N_3(v, w) \subseteq N(w)$ and $N_3(v, w) \not\subseteq N(v)$, remove $N_3(v, w)$ and $N_2(v, w) \cap N(w)$ from G and add a new white vertex w' and the edge (w, w') to G.

Case 2. If $N_3(v, w)$ cannot be dominated by a single vertex in $\{v, w\}$, then remove $N_2(v, w) \cup N_3(v, w)$ from G and add two new white vertices v', w' and the edges (v, v'), (w, w') to G.

Rule 3. For each black vertex v in G, if there exists a black vertex $x \in N_2(v) \cup N_3(v)$, color x white, and remove the edges between x and all other white vertices in G.

Rule 4. For every two black vertices v and w, if $N_3(v, w) \neq \emptyset$, then for every black vertex $x \in N_2(v, w) \cup N_3(v, w)$ that does not dominate all vertices in $N_3(v, w)$, color x white and remove all the edges between x and the other white vertices in G.

Rule 5. For every quasi-simple region $R = R(v, w)$ between two vertices v and w, if v is black, then for every black vertex $x \in N_2(v, w) \cup N_3(v, w)$ strictly inside R that does not dominate all vertices in $N_2(v, w) \cup N_3(v, w)$ strictly inside R, color x white and remove all the edges between x and the other white vertices in G.

Rule 6. For every two white vertices u and v, if $N(u) \subseteq N(v)$, and $u \in N_2(w) \cup N_3(w)$ for some black vertex w, then remove v.

Rule 7. For every black vertex v, if every vertex $u \in W(v)$ is connected to all the vertices in $B(v)$, then remove all the vertices in $W(v)$ from G.

Rule 8. For every two black vertices v and w, let $W(v,w) = W(v) \cap W(w)$. If $|W(v,w)| \geq 2$ and there is a degree-2 vertex $u \in W(v,w)$, then remove all vertices in $W(v,w)$ except u, add a new degree-2 white vertex u', and connect u' to both v and w.

A graph G is said to be *reduced* if every vertex in G is colored white or black, and the application of **Rules 1–8** leaves the graph G unchanged. That is, the application of any of the above rules does not change the color of any vertex in G, nor does it change the structure of G.

Theorem 2. *Let G be a graph with n vertices. Then in time $O(n^3)$ we can construct a graph G' from G such that: (1) G' is reduced, (2) $\gamma(G') = \gamma(G)$,* (3) *there exists a minimum dominating set for G' that excludes all white vertices of G', and* (4) *from a minimum dominating set for G' a minimum dominating set for G can be constructed in linear time.*

5 A problem kernel for PLANAR DOMINATING SET

Let G be a reduced graph, and let D be a minimum dominating set for G consisting of black vertices such that $|D| = k$. In this section, we will show that the number of vertices n in G is bounded by $67k$. The following definitions are adopted from [1]. The reader is referred to [1] for more details.

Given any dominating set D in a graph G, a *D-region* decomposition of G is a set $\Re$ of regions between pairs of vertices in D such that:

1. For any region $R = R(v,w)$ in $\Re$, no vertex in D is in $V(R)$. That is, a vertex in D can only be an endpoint of a region in $\Re$.
2. No two distinct regions R_1, $R_2 \in \Re$ intersect. However, they may touch each other by having common boundaries.

Note that all the endpoints of the regions in a D-region decomposition are vertices in D. For a D-region decomposition $\Re$, define $V[\Re] = \bigcup_{R \in \Re} V[R]$. A D-region decomposition is *maximal*, if there is no region R such that $\Re' = \Re \cup R$ is a D-region decomposition with $V[\Re] \subsetneq V[\Re']$.

For a D-region decomposition $\Re$, associate a planar graph $G_\Re(V_\Re, E_\Re)$ with possible multiple edges, where $V_\Re = D$, and such that there is an edge between two vertices v and w in $G_\Re$ if and only if $R(v,w)$ is a region in $\Re$. A planar graph with multiple edges is called *thin*, if there is a planar embedding of the graph such that for any two edges e_1 and e_2 between two distinct vertices v and

w in the graph, there must exist two more vertices which sit inside the disjoint areas of the plane enclosed by e_1 and e_2.

Alber et al. [1] showed that the number of edges in a thin graph of n vertices is bounded by $3n-6$. They also showed that for any reduced plane graph G and a dominating set D of G, there exists a maximal D-region decomposition for G such that $G_{\Re}$ is thin. Since the maximal D-region decomposition in [1] starts with any dominating set D and is not affected by the color a vertex can have, the same results in [1] hold true for our reduced graph G whose vertices are colored black/white, and with a minimum dominating set D consisting only of black vertices. The above discussion is summarized in the following proposition.

Proposition 1. *Let G be a reduced graph and D a dominating set of G consisting of black vertices. Then there exists a maximal D-region decomposition $\Re$ of G such that $G_{\Re}$ is thin.*

Corollary 1. *Let G be a reduced graph with a minimum dominating set D consisting of k black vertices, and let $\Re$ be a maximal D-region decomposition of G such that $G_{\Re}$ is thin. Then the number of regions in $\Re$ is bounded by $3k-6$.*

Proof. *The number of regions in $\Re$ is the number of edges in $G_{\Re}$. Since $G_{\Re}$ has $|D| = k$ vertices, by [1], the number of edges in $G_{\Re}$ is bounded by $3k-6$.*

In the remainder of this section, $\Re$ will denote a maximal D-region decomposition of G such that $G_{\Re}$ is thin. Let u and v be two vertices in G. We say that u and v are *boundary-adjacent* if (u, v) is an edge on the boundary of some region $R \in \Re$. For a vertex $v \in G$, denote by $N^*(v)$ the set of vertices that are boundary-adjacent to v. Note that for a vertex $v \in D$, since v is black, by **Rule 3**, all vertices in $N_2(v) \cup N_3(v)$ must be white.

Proposition 2. *Let $v \in D$. The following are true.*

(a) (Lemma 6, [1]) Every vertex $u \in N_1(v)$ is in $V[\Re]$.
(b) The vertex v is an endpoint of a region $R \in \Re$. That is, there exists a region $R = R(x, y) \in \Re$ such that $v = x$ or $v = y$.
(c) Every vertex $u \in N_2(v)$ which is not in $V[\Re]$ is connected only to v and to vertices in $N^*(v)$.

Let x be a vertex in G such that $x \notin V[\Re]$. Then by part (b) in Proposition 2, $x \notin D$. Thus, $x \in N(v)$ for some black vertex $v \in D \subseteq V[\Re]$. By part (a) in Proposition 2, $x \notin N_1(v)$, and hence, $x \in N_2(v) \cup N_3(v)$. By **Rule 3**, the color of x must be white. Let $R = R(v, w)$ be a region in $V[\Re]$ of which v is an endpoint (such a region must exist by part (b) of Proposition 2). We distinguish two cases.

Case A. $x \in N_3(v)$. Since v is black, by **Rule 1**, this is only possible if $deg(x) = 1$ and $N_2(v) = \emptyset$ (in this case x will be the white vertex added by the rule). In such case it can be easily seen that we can flip x and place it inside R without affecting the planarity of the graph.

Case B. $x \in N_2(v)$. Note that in this case $N_3(v) = \emptyset$, and x is only connected to v and $N^*(v)$ by part (c) in Proposition 2. If $deg(x) = 2$, by a similar argument to **Case A** above, x can be flipped and placed inside R.

According to the above discussion, it follows that the vertices in G can be classified into two categories: (1) those vertices that are in $V[\Re]$; and (2) those that are not in $V[\Re]$, which are those vertices of degree larger than two that belong to $N_2(v)$ for some vertex $v \in D$, and in this case must be connected only to vertices in $N^*(v)$. To bound the number of vertices in G we need to bound the number of vertices in the two categories. We start with the vertices in category (2).

Let O denote the set of vertices in category (2). Note that all vertices in O are white, and no two vertices u and v in O are such that $N(u) \subseteq N(v)$. To see why the latter statement is true, note that every vertex in O must be in $N_2(w)$ for some black vertex $w \in D$. So if $N(u) \subseteq N(v)$, then by **Rule 6**, v would have been removed from the graph. To bound the number of vertices in O, we will bound the number of vertices in O that are in $N_2(v)$ where $v \in D$. Let us denote this set by $N^\dagger(v)$. Let $N^*_\dagger(v)$ be the set of vertices in $N^*(v)$ that are neighbors of vertices in $N^\dagger(v)$. Note that every vertex in $N^\dagger(v)$ has degree ≥ 3, is connected only to v and to $N^*_\dagger(v)$, and no two vertices x and y in $N^\dagger(v)$ are such that $N(x) \subseteq N(y)$.

Proposition 3. $|N^\dagger(v)| \leq 3/2|N^*_\dagger(v)|$.

Lemma 1. *The number of vertices in category (2) (i.e., the number of vertices not in $V[\Re]$) is bounded by $18k$.*

Proof. Let v and w be any two distinct vertices in D and observe the following. First, $N^\dagger(v) \cap N^\dagger(w) = \emptyset$, because if $u \in N^\dagger(v) \cap N^\dagger(w)$ then (v, u, w) would be a degenerated region with $u \notin V[\Re]$ contradicting the maximality of $\Re$. Second, from the first observation it follows that $w \notin N^*_\dagger(v)$ and $v \notin N^*_\dagger(w)$(in general no vertex $a \in D$ belongs to $N^*_\dagger(b)$ for any vertex $b \in D$); otherwise, there exists a vertex $u \in N^\dagger(v)$ that is connected to w, and hence u $N^\dagger(v) \cap N^\dagger(w)$, contradicting the first observation. Third, $N^*_\dagger(v) \cap N^*_\dagger(w) = \emptyset$; otherwise, there exists a vertex $u \in N^*_\dagger(v) \cap N^*_\dagger(w)$ that is connected to a category-(2) vertex $a \in N^\dagger(v)$ (or $b \in N^\dagger(w)$) and the degenerated region (v, a, u, w) (or (w, b, u, v)) would contain the vertex $a \notin \Re$ (or $b \notin \Re$), contradicting the maximality of $\Re$.

Let B be the number of vertices not in D that are boundary-adjacent to vertices in D (i.e., in $N^*(v) - D$ for some $v \in D$). Combining the above observations with Proposition 3, it follows that the number of category-(2) vertices is

$$\sum_{v \in D} |N^\dagger(v)| \leq \frac{3}{2} \sum_{v \in D} |N^*_\dagger(v)| \leq 3B/2$$

According to the definition of a region, each region in $\Re$ has at most six vertices on its boundary two of which are vertices in D. Thus, each region in $\Re$ can contribute with at most four vertices to B. By Corollary 1, the number of *regions in $\Re$ is bounded by $3k-6$. It follows that $B \leq 12k-24$, and hence, the number of category-(2) vertices is bounded by $18k-36 < 18k$. This completes the proof.*

To bound the number of vertices in category (1), fix a region $R(v,w)$ between $v, w \in D$. We have the following lemma.

Lemma 2. *Let $R = R(v,w)$ be a region in $V[\Re]$. The number of vertices in $V(R)$ is bounded by 16.*

Theorem 3. *The number of vertices in the reduced graph G is bounded by $67k$.*

Proof. *By Lemma 1, the number of category-(2) vertices in G is bounded by $18k$. According to the discussion before, if we use the $18k$ upper bound on the number of category-(2) vertices, then we can assume that each region in $\Re$ is nice (if this is not the case we obtain a better upper bound on the total number of vertices in G). By Corollary 1, the number of regions in $\Re$ is bounded by $3k-6$. According to Lemma 2, the number of vertices in $V(R)$, where $R \in \Re$ is a nice region, is bounded by 16. It follows that the number of vertices in $V(\Re)$ is bounded by $48k-96$. Thus, the number of vertices in $V[\Re]$, and hence, in category (1), is bounded by $48k-96$ plus the number of vertices in D which are the endpoints of the regions in $\Re$. Therefore the number of vertices in $V[\Re]$ is bounded by $49k-96$, and the total number of vertices in G is bounded by $67k-96 < 67k$. This completes the proof.*

Theorem 4. *Let G be a planar graph with n vertices. Then in time $O(n^3)$, computing a dominating set for G of size bounded by k can be reduced to computing a dominating set of size bounded by k, for a planar graph G' of $n' < n$ vertices, where $n' \leq 67k$.*

Proof. *According to Theorem 2, in time $O(n^3)$ we can construct a reduced graph G' from G where $\gamma(G') = \gamma(G)$, and such that a dominating set for G can be constructed from a dominating set for G' in linear time. Moreover, the graph G' has no more than n vertices. If G has a dominating set of size bounded by k, then G' has a dominating set of size bounded by k (since $\gamma(G) = \gamma(G')$), and by Theorem* **??***, we must have $n' \leq 67k$, so we can work on computing a dominating set for G'. If this is not the case, then G does not have a dominating set of size bounded by k, and the answer to the input instance is negative. This completes the proof.*

Theorem 4, together with Theorem 1, gives:

Corollary 2. *For any $\epsilon > 0$, there is no $(67/66 - \epsilon)k$ kernel for* PLANAR NON-BLOCKER.

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