

A Sum Labelling for the Flower $f_{q,p}$ ¹

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Abstract

We provide an optimal sum labelling scheme for the generalised friendship graph, also known as the flower (a symmetric collection of cycles meeting at a common vertex) and show that its sum number is 2.

Keywords: Sum labelling, friendship graphs, flower graphs.

1 Introduction

All graphs considered in this paper are simple, finite and undirected.

A sum labeling is a mapping λ from the vertices of G into the positive integers such that, for any two vertices $u, v \in V(G)$ with labels $\lambda(u)$ and $\lambda(v)$, respectively, (uv) is an edge if and only if $\lambda(u) + \lambda(v)$ is the label of another vertex in $V(G)$. Any graph supporting such a labeling is called a *sum graph*. Sum graphs are necessarily disconnected, so in order to sum label a connected graph it became necessary to add (as a disjoint union) a further component. By convention this disconnected component is a set of isolated vertices known as *isolates* and the labeling scheme that requires the fewest isolates is termed *optimal*. The number of isolates required for a graph to support a sum labelling is known as the *sum number* of the graph.

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Graph Family G	Sum Number $\sigma(G)$	Reference
Tree T_n	1	[2]
Cycle C_n	$2 \ n \neq 4,$	[3]
4-Cycle C_4	3	[3]
Even Wheel W_n	$\frac{n}{2} + 2, \ n \text{ even}$	[5, 9]
Odd Wheel W_n	$n + 2, \ n \text{ odd}$	[5, 9]
Complete Graph K_n	$2n - 3$	[1]
Cocktail Party Graph $H_{2,n}$	$4n - 5$	[8]
Complete Bipartite Graph $K_{m,n}, m < n$	$\left\lceil k \frac{n-1}{2} + \frac{m}{k-1} \right\rceil$ where $k = \left\lceil \frac{\sqrt{1+(8m+n-1)(n-1)}}{2} \right\rceil$	[4, 6, 10, 11]

Table 1: Sum Numbers of Some Well Known Graphs

Sum labeling of graphs was introduced by Harary [3] in 1990 and since that time the problem of finding an optimal labeling for a family of graphs has been shown to be difficult, even for fairly simple graphs. Table 1 lists the sum numbers for some well known graphs.

A lower bound for the sum number of a connected graph is the minimum degree (δ) of the graph. Graphs achieving this bound are known as δ -optimal [7]. In this paper we introduce a new class of connected graphs and provide a scheme for imposing on them a δ -optimal labeling.

The motivation for this paper is the curious case of the 4-cycle. As shown in Table 1, all cycles C_n are optimal summable except C_4 . One method of investigation was to consider two 4-cycles with a common vertex, which was discovered to be optimal summable. From there the natural extension was to consider k 4-cycles with a common vertex, then k n -cycles with a common vertex.

The *friendship graph* f_n is a collection of n triangles with a common vertex. It may be also pictured as a wheel with every alternate rim edge removed, see Figure 1. The *generalised friendship graph* $f_{q,p}$ is a collection of p cycles (all of order q), meeting at a common vertex (see Figure 3). In this section we will refer to the friendship graph f_n as an instance of the

generalised friendship graph and write it as $f_{3,n}$. The generalised friendship graph is, because of its shape, also referred to as a *flower*. In this nomenclature the cycles are referred to as petals.

For our purposes, we refer to vertices in the following way: the central vertex is named c and all other vertices addressed in the form v_j^i , where $i, 1 \leq i \leq p$ indicates which cycle contains the vertex and $j, 1 \leq j \leq q$ points to the position of the vertex within the cycle. So $c = v_q^i$ for all i .

To develop our labeling scheme we first consider the cases $q = 3, 4, 5$ before proving the result for the general case by considering both odd and even cycles.

2 Constructing a sum labelling for $f_{q,p}$

Lemma 1 $f_{3,p}$ is optimal summable.

Proof: Begin a sum labelling for $f_{3,p}$ by labelling the centre vertex $c = 1$. Commencing from any triangle, distribute the labels $a, a + 1, a + 2, \dots$ clockwise about the centre, one label for each triangle. The maximum label at this stage is $a + (p - 1)$ on the p^{th} triangle. Then continue the labelling sequence, distributing the labels $a + p, a + (p + 1), \dots$ counterclockwise from the p^{th} triangle. The maximum label is now $a + (2p - 1)$ and is adjacent to the label a . Now all of the edges independent of the centre join vertices whose combined sum is $2a + (2p - 1)$ and the only edge not yet accounted for in the labelling is the edge between the maximum label and the centre (whose vertex sums add to $a + 2p$).

Since both $2a + (2p - 1)$ and $a + 2p$ are larger than the maximum label in the graph, they must be (the only required) isolates. It only remains to ensure that no further edges are induced. Setting a large enough (say $a = 2p$) suffices. \square

Lemma 2 $f_{4,p}$ is optimal summable.

Proof: Give the centre vertex the label 5 and label the outer vertices according to the scheme,

$$\begin{aligned} v_1^1 &= 8, v_2^1 = 1, v_3^1 = 2^{p-3} \times 50 - 2^{p-1} - 1 \\ v_1^2 &= 2^{p-3} \times 50 - 2^{p-1} - 10, v_2^2 = 9, v_3^2 = 2^{p-3} \times 50 - 2^{p-1} - 5 \\ v_1^i &= v_2^i - 5, v_2^i = 2^{i-4} \times 50 - 2^{i-2}, v_3^i = v_2^i - 10. \end{aligned}$$

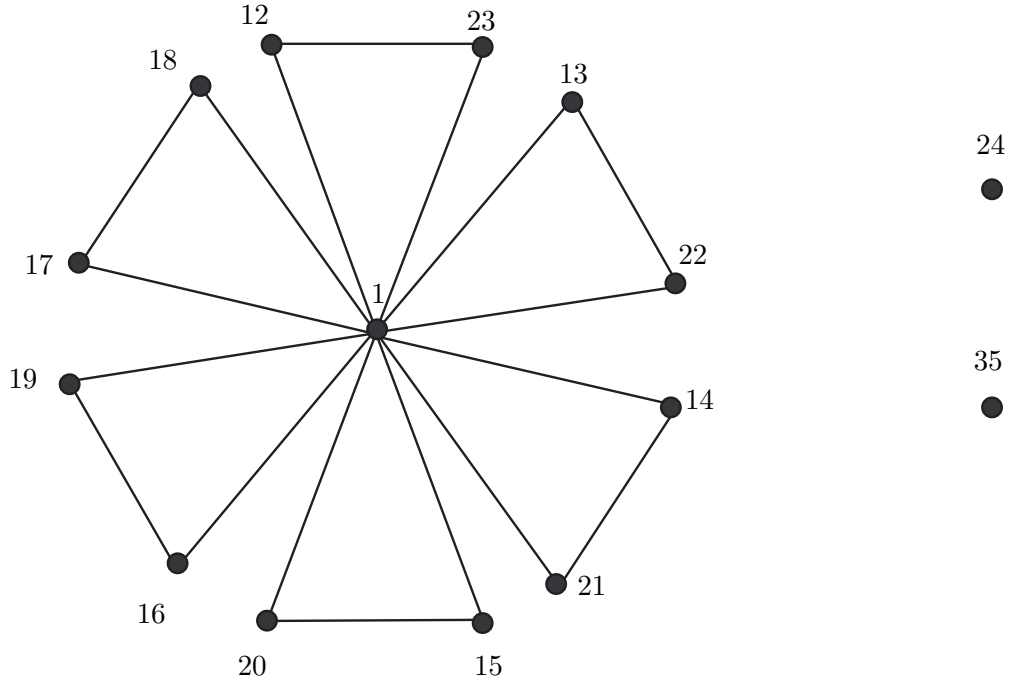


Figure 1: The flower $f_{3,6}$ and its labeling.

This labelling provides for 2 isolates (x, y) of the form

$$x = 2^{p-3} \times 50 - 2^{p-1}, y = 2^{p-3} \times 50 - 2^{p-1} + 4.$$

Simple arithmetic verifies that the edges on the first two cycles are witnessed by the labelling. The following equations show that all edges of intermediate cycles are witnessed.

$$\begin{aligned} v_1^i &= v_3^i + 5 = v_1^{i-1} + v_2^{i-1} \\ v_2^i &= v_1^i + 5 \\ v_3^i &= v_3^{i-1} + v_2^{i-1} \end{aligned}$$

The relationship $v_2^i = 2 \times v_2^{i-1}$ ensures that the labels on the individual cycles are sufficiently well spaced so as to prevent edges being induced between cycles. \square

Lemma 3 $f_{5,p}$ is optimal summable.

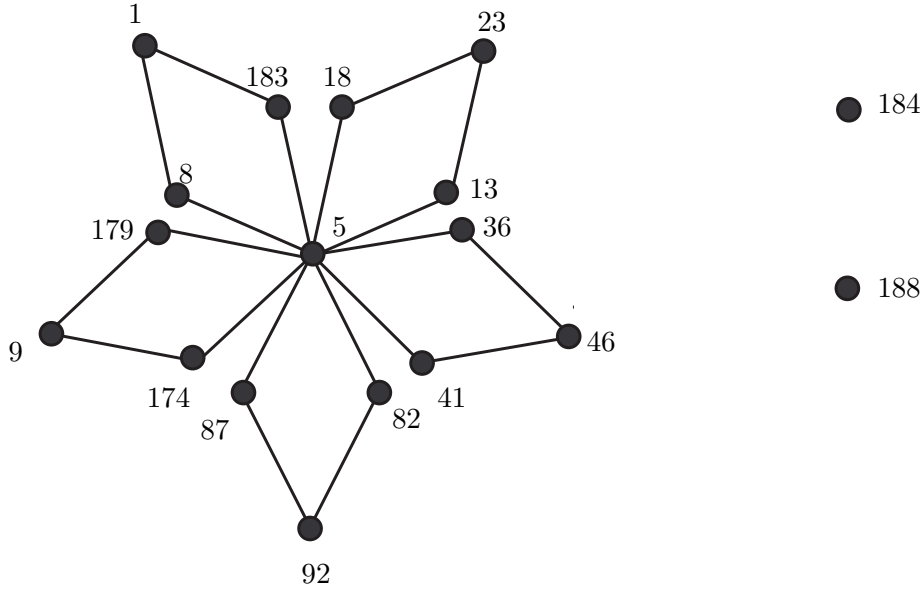


Figure 2: The flower $f_{4,5}$ and its labeling.

Proof: Label the centre $c = 1$ and the vertices on the first petal by

$$v_1^1 = 3, v_2^1 = 5, v_3^1 = 8, v_4^1 = 4$$

and similarly for subsequent petals (except for the last) following the scheme

$$\lambda(v_1^i) = \lambda(v_3^{i-1}) + \lambda(v_4^{i-1})$$

$$\lambda(v_2^i) = \lambda(v_4^i) + c$$

$$\lambda(v_3^i) = \lambda(v_1^i) + \lambda(v_2^i)$$

$$\lambda(v_4^i) = \lambda(v_1^i) + c.$$

Note that every vertex is working except for the two smallest and that the multiplicity of v_4^i is 2

$$\lambda(v_4^i) = \lambda(v_1^i) + c = \lambda(v_2^{i-1}) + \lambda(v_3^{i-1}).$$

The last cycle is labelled conventionally from v_1^i to v_4^i to avoid inducing any extra edges. More formally,

$$\lambda(v_1^p) = \lambda(v_3^{p-1}) + \lambda(v_4^{p-1})$$

$$\begin{aligned}\lambda(v_2^p) &= \lambda(v_1^p) + c \\ \lambda(v_3^p) &= \lambda(v_1^p) + \lambda(v_2^p) \\ \lambda(v_4^p) &= \lambda(v_2^p) + \lambda(v_3^p)\end{aligned}$$

leaving just two isolates to account for the edges adjacent to the vertex with the highest label(v_4^p).

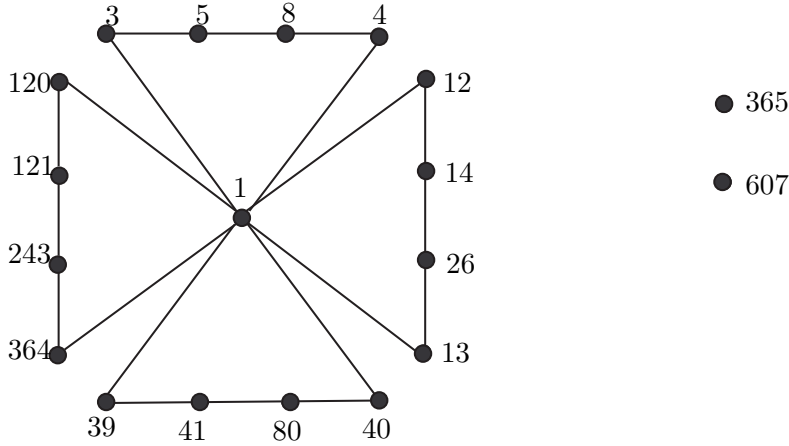


Figure 3: An optimal sum labelling of the graph $f_{5,4}$.

That no further edges are induced can best be shown using an explicit formula for the labelling. For the i^{th} cycle, an alternative description of the labelling is as follows.

$$\begin{aligned}\lambda(v_1^i) &= 2 \cdot 3^{i-1} + \frac{1}{2}(3^i - 1) - 1 \\ \lambda(v_2^i) &= \lambda(v_1^i) + 2 \\ \lambda(v_3^i) &= 2 \cdot \lambda(v_2^i) \\ \lambda(v_4^i) &= \lambda(v_1^i) + 1\end{aligned}$$

These formulae keep any "unwanted" edges being induced within a petal and the exponents of 3 forbid the inducing of any extra edges between the petals. \square

Lemma 4 *The class of graphs $f_{q,p}$, $q > 5$, q odd is optimal summable.*

Proof: We begin by labelling the centre vertex c with the label $\lambda(c) = 2p(q-3)$. Before labelling

the remaining vertices we define two parameters,

$$\begin{aligned}\gamma &\equiv i \pmod{2} \\ \delta &\equiv j \pmod{2}.\end{aligned}$$

The labelling λ may now be defined as

$$\begin{aligned}\lambda(v_j^i) &= \gamma(\delta(c + 2(j-1)p + (2i-1)) - (\delta-1)(c - 2(j-4)p - (2i-1))) \\ &\quad - (\gamma-1)(\delta(2(j-1)p + (2i-1)) - (\delta-1)(c - 2(j-1-q)p - (2i-1)))\end{aligned}$$

Simple substitution into the above formula shows that,

$$\begin{aligned}\lambda(v_j^i) + \lambda(v_{j+1}^i) &= 2c + 4p, & j \text{ odd} \\ \lambda(v_j^i) + \lambda(v_{j+1}^i) &= 2c + 8p, & j \text{ even}\end{aligned}$$

Let $2c + 4p = x$ and $2c + 8p = y$. Then since $y > x > \max \lambda(v_j^i)$, x and y must both be isolates. When considering the edges adjacent to c it is also a simple matter of substitution to see that, for each i ,

$$\begin{aligned}\lambda(v_1^i) + c &= \lambda(v_{q-2}^i) \text{ and,} \\ \lambda(v_{q-1}^i) + c &= \lambda(v_2^i),\end{aligned}$$

and so, with just two isolates we are able to identify all edges. It remains to check that the labelling has induced no extra edges.

It is clear that there are no induced edges between the only three vertices labelled with even integers (c, x and y). Inspection shows us that all other vertices are adjacent to an edge with sum number x so there are only three conditions to be checked;

1. there is no vertex z such that $\lambda(z) + \lambda(v_j^i) = y$, for $j = 1, q-1$,
2. there is no vertex z such that $\lambda(z) + \lambda(v_j^i) = c$,
3. there is no vertex v_j^i such that $c + \lambda(v_j^i) = \lambda(v_{j'}^i)$ where $j \neq 1, q-1$.

That these three conditions are realised by the labelling may be seen by considering the set of labels as the two arithmetic sequences with common difference 4 as shown below.

$$\begin{aligned}\lambda(v_1^2) &= 3, \dots, x - c - 1 = \lambda(v_2^1) \\ \lambda(v_1^1) &= c + 1, \dots, x - 3 = \lambda(v_2^2)\end{aligned}$$

The first sequence consists of numbers congruent to 3 mod 4 with the maximum value of $c + 4p - 1$ and the second sequence consists of numbers congruent to 1 mod 4 with the minimum value of $c + 1$. The first condition holds since there is no label large enough in the first sequence or small enough in the second sequence to label z . The other conditions also hold since adding c to any term in the first sequence gives it a value greater than the maximum term in that sequence. \square

As an example Figure 4 shows a labelling of $f_{7,4}$.

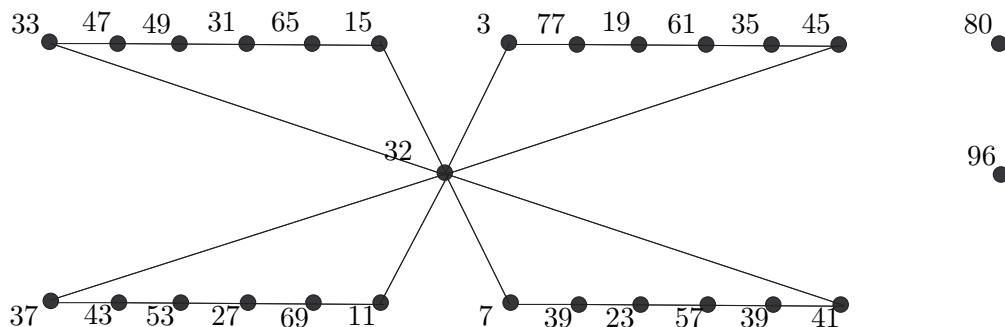


Figure 4: A sum numbering for $f_{7,4}$.

Lemma 5 *The class of graphs $f_{q,p}$, $q > 4$, q even is optimal summable.*

Proof: Begin by labelling the centre $\lambda(c) = 2p(q - 2)$ and as before let $\delta \equiv j \pmod{2}$. The labelling for the remaining vertices is

$$\lambda(v_j^i) = \delta(2i - 1 + 2p(j - 1)) + (1 - \delta)(3c - (2i - 1) - 2p(j - 1)).$$

This results in isolates

$$x = 3c - 2p$$

$$y = 3c + 2p$$

each accounting for alternate edges independent from c . For the edges adjacent to c , it is easy to see that

$$\begin{aligned} v_1^i + c &= v_{q-1}^i \\ v_{q-1}^i + c &= v_{q-2}^{1+p-i}. \end{aligned}$$

This time we can check the correctness of the labelling scheme by noting that since all labels on the cycles (except the centre) are odd, no cycle vertex can induce an ‘unwanted’ edge. Further the labels are well spaced enough that only vertices with labels $\lambda(v_{q-1}^i)$ and $\lambda(v_{q-2}^i)$ are working vertices (apart from the isolates) and they witness edges $(c, \lambda(v_1^i))$ and $(c, \lambda(v_{q-1}^{p-i+1}))$ respectively. \square

As an example we offer a labelling of $f_{8,4}$. Following the scheme we have $c = 48, x = 136, y = 152$ and the cycles (petals) labelled as in Figure 5

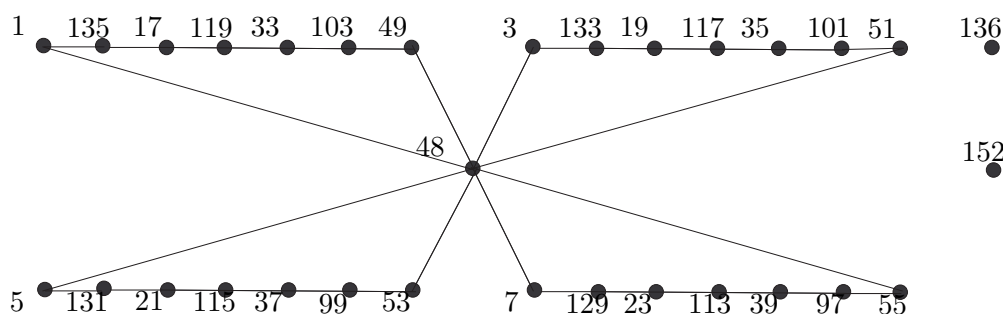


Figure 5: A sum numbering for $f_{8,4}$.

These five lemmas together give us,

Theorem 1 *The generalised friendship graph $f_{q,p}$ has sum number 2.*

Let us conclude with a peculiar remark on the complexity (size) of the labelling: We succeeded in finding a *polynomial-size (optimal) sum labelling* of all flowers besides $f_{4,p}$ and $f_{5,p}$, where the label numbers grow exponentially with the size of the flowers. Open question: Is there a “smaller” labelling for the mentioned particular cases?

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