Constructive aspects of analytic functions

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1 Power Series and Analytic Functions

Power series and their sum functions ('analytic functions') are a central topic in the field of (complex) analysis. Obviously they should also be thoroughly examined in *constructive* analysis.

The following two classical results are the basis for our investigation:

> (A) Consider a power series $\sum_{k=0}^{\infty} a_k \cdot z^k$ (with complex numbers a_k and z) converging absolutely for an arbitrary $z \neq 0$.
>
> Then there is a function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ such that $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$ in a open neighborhood of zero. f is differentiable (in $\mathbb{C}$) on this neighborhood.

and

> (B) Consider a function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ that is differentiable in a (complex) neighborhood of zero. Then there is a sequence$(a_k)_{k\in\mathbb{N}}$ of complex numbers such that $f(z) = \sum_{k=0}^{\infty} a_k \cdot z^k$ in a neighborhood of zero.

So, a natural and immediate question is: Which of the following mappings are constructive / computable / efficient?

$$(a_k)_{k\in\mathbb{N}} \mapsto f \qquad\qquad f \mapsto (a_k)_{k\in\mathbb{N}}$$

There are several results concerning this topic ([KoFr88, Mu87], see also [Sch90, Ko91, Mu93]), but they are primarily regarding computational complexity.

As these papers do not contain definitions of computability for operators, the formulations of the results are inherently non-constructive, e.g. '*If f is analytic and polynomial-time computable, then the corresponding sequence (a_k) is also polynomial-time computable.*' Nevertheless, the proofs are almost constructive.

In the following we will examine power series from a constructive viewpoint. Basis will be [Mu93].

2 Representations

For a precise examination of constructivity and effectivity we need proper models, especially for proofs of non-constructivity. The following will be based on the theory of representations [We87]. We will omit many details and concentrate on one central result.

Essentially each object of a set M is viewed as the result of an approximating process in a 'natural' topology τ_M. As starting point, we will use a countable subbase $\mathbb{S}$ of τ_M. In order to examine computability or complexity, $\mathbb{S}$ should have an accepted model of computability or complexity.

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For $\mathbb{R}$, $\mathbb{S}$ can e.g. be chosen as the set of open intervals with rational borders. To study complexity, the set of intervals of the form $O_{n,d} = (d - 2^{-n}, d + 2^{-n})$ for $n \in \mathbb{N}$ and $d \in \mathbb{D} := \{m \cdot 2^e \mid m, e \in \mathbb{Z}\}$ ('dyadic numbers') is preferable.

Consider a function $\sigma : \mathbb{N} \to \mathbb{S}$ (representing an approximating process). Often, we will be able to assign an $x \in M$ as a 'limit' to this process σ. If the derived mapping $\varrho : (\mathbb{N} \to \mathbb{S}) \dashrightarrow M$ is surjective, we say that ϱ is a representation of M and that σ is a name for $\varrho(\sigma)$.

Computability of functions Γ from $(\mathbb{N} \to \mathbb{S}_1)$ to $(\mathbb{N} \to \mathbb{S}_2)$ is defined using oracle Turing machines, enumeration operators or similar concepts. Given representations $\varrho_i : (\mathbb{N} \to \mathbb{S}_i) \dashrightarrow M_i$, we may derive computability of functions between M_1 and M_2: $f \colon M_1 \dashrightarrow M_2$ is computable iff there is a computable Γ such that $f\varrho_1(\sigma) = \varrho_2\Gamma(\sigma)$ as soon as $f\varrho_1(\sigma)$ is defined.

As a generalization of computability, continuous functions between $(\mathbb{N} \to \mathbb{S}_1)$ and $(\mathbb{N} \to \mathbb{S}_2)$ can be defined in a natural way. Essential for the theory of representations is that a function $f : M_1 \dashrightarrow M_2$ is (or better: may be called) *constructive*, iff there is a *continuous* Γ such that $f\varrho_1(\sigma) = \varrho_2\Gamma(\sigma)$ as soon as $f\varrho_1(\sigma)$ is defined. For a motivation, see e.g. [We87, We94, We95]. Please note the implicit dependence on the representations and the topology.

A central result of the theory of representations can be formulated as follows: Suppose the topologies τ_{M_i} induced by $\mathbb{S}_i$ are T_0-topologies and consider 'standard' representations ϱ_i defined by $\varrho_i(\sigma) = x$ iff $\{\sigma(n) \mid n \in \mathbb{N}\}$ is a subbase of the set of neighborhoods of x. Then a function $f : M_1 \dashrightarrow M_2$ is constructive (in the sense above) if and only if f is continuous w.r.t. τ_{M_1} and τ_{M_2}!

The identity between constructivity and continuity is also valid for 'admissible' representations, i.e. representations that are equivalent to such 'standard' representations.

In our case, three essentially different spaces are of interest:

Real numbers: As said before, the intervals $O_{n,d} := (d - 2^{-n}, d + 2^{-n})$ for $n \in \mathbb{N}$ and for dyadic numbers $d \in \mathbb{D} := \{m \cdot 2^e \mid m, e \in \mathbb{Z}\}$ are an appropriate subbase. The corresponding topology is the natural topology on $\mathbb{R}$.

An admissible representation $\varrho_{\mathbb{R}}$ can be defined as follows: $\varrho_{\mathbb{R}}(\sigma) := x$ iff for any n there is a d_n such that $\sigma(n) = O_{n,d_n}$ and $x \in \sigma(n)$, so d_n approximates x with an error less than 2^{-n}.

Using a binary notation of the dyadic numbers, $\varrho_{\mathbb{R}}$ also induces a definition of complexity of real numbers that is quite natural, and furthermore, $\varrho_{\mathbb{R}}$ can be interpreted as a formalization of a computer arithmetic using floating point numbers with varying precision.

Sequences of numbers: The usual definitions for computability of sequences $(a_k)_{k\in\mathbb{N}} \in S(\mathbb{R}) := \mathbb{R}^\infty$ are based on approximations $d_{k,n}$ such that $|d_{k,n} - a_k| \le 2^{-n}$.

So here we should use the subbase $\{S_{k,n,d} \mid k, n \in \mathbb{N},\ d \in \mathbb{D}\}$, where $S_{k,n,d} := \mathbb{R}^k \times (d - 2^{-n}, d + 2^{-n}) \times \mathbb{R}^\infty$. Hence $S_{k,n,d}$ gives us an approximation for just the k-th element of the sequence.

The induced topology $\tau_{S(\mathbb{R})}$ is obviously of type T_0: If $(a_k)_{k\in\mathbb{N}} \neq (b_k)_{k\in\mathbb{N}}$, then $a_k \neq b_k$ for some $k \in \mathbb{N}$. Let n, d be such that $a_k \in O_{n,d}$ but $b_k \notin O_{n,d}$. Then $S_{k,n,d}$ is a neighborhood of (a_k) but not of (b_k).

An admissible representation $\varrho_{S(\mathbb{R})}$ can be defined by $\varrho_{S(\mathbb{R})}(\sigma) = (a_k)_{k\in\mathbb{N}}$ iff for any n, k there is a $d_{k,n} \in \mathbb{D}$ such that $\sigma(<n,k>) = O_{k,n,d_{k,n}}$ and $a_k \in \sigma(<n,k>)$, so $d_{k,n}$ approximates a_k with an error less than 2^{-n}. Here $<n,k>$ denotes the standard bijective pairing function on $\mathbb{N}^2$.

Functions: Computability of real functions f is usually defined by computable enumerations of pairs $(O_{n,d}, O_{m,e})$ of intervals such that $fO_{n,d} \subseteq O_{m,e}$ with the additional property that for any $x \in dom(f)$ and any $m \in \mathbb{N}$ there must be a pair $(O_{n,d}, O_{m,e})$ with $x \in O_{n,d}$ (implying continuity of f).

Let $C_{n,d} := [d - 2^{-n}, d + 2^{-n}]$. Then an equivalent definition may be based on pairs $(C_{n,d}, O_{m,e})$ with $fC_{n,d} \subseteq O_{m,e}$, where for $x \in dom(f)$ and $m \in \mathbb{N}$ now there must be a pair $(C_{n,d}, O_{m,e})$ such that $x \in O_{n,d}$ still holds.

So for any $x \in C_{n,d}$, the dyadic number e approximates $f(x)$ with an error less than 2^{-m}.

Now for any pair $(C_{n,d}, O_{m,e})$ let $FP_{n,d,m,e}$ be the set of all continuous partial functions $f : \mathbb{R} \dashrightarrow \mathbb{R}$ satisfying $fC_{n,d} \subseteq O_{m,e}$.

Unfortunately, the induced topology is not T_0! For example, functions like $f_1 : (0,1) \cup (1,2) \mapsto 1$ and $f_2 : (0,2) \mapsto 1$ share the same neighborhoods. Furthermore, the cardinality of the set $CP(\mathbb{R})$ of all continuous partial functions is higher than that of the continuum. So there is no surjective function from $\mathbb{N} \to \mathbb{S}$ to $CP(\mathbb{R})$, as long as the subbase $\mathbb{S}$ must be countable.

So we are bound to use just a subset of $CP(\mathbb{R})$: Often the sets $C(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ or $C([0,1]) = \{f : [0,1] \to \mathbb{R}\}$ are used. Both choices are not appropriate, because analytic functions usually are partial with varying domains. Furthermore, if f is analytic in x, then there is a neighborhood $\mathcal{U}$ of x such that f (or a continuation of f) is analytic on $\mathcal{U}$. So representations based on open sets seem to be a better choice.

Consider the set $CO(\mathbb{R})$ of all continuous partial functions $f : \mathbb{R} \dashrightarrow \mathbb{R}$ with *arbitrary open* domains $dom(f)$. Let $FO_{n,d,m,e}$ be the set of all functions $f \in CO(\mathbb{R})$ satisfying $C_{n,d} \subseteq dom(f)$ and $fC_{n,d} \subseteq O_{m,e}$ (for $n, m \in \mathbb{N}$, $d, e \in \mathbb{D}$).

2.1 *Theorem.* *The topology $\tau_{CO(\mathbb{R})}$ induced by $\{FO_{n,d,m,e} : n, m \in \mathbb{N}, d, e \in \mathbb{D}\}$ is a T_0-topology, so the following representation $\varrho_{CO(\mathbb{R})}$ is admissible: $\varrho_{CO(\mathbb{R})}(\sigma) := f$ iff $\{\sigma(n) : n \in \mathbb{N}\} = \{FO_{n,d,m,e} : C_{n,d} \subseteq dom(f) \wedge fC_{n,d} \subseteq O_{m,e}\}$.* □

Proof: Consider $f, g \in CO(\mathbb{R})$ with $f \neq g$. Then there is an $x \in \mathbb{R}$ such that either $x \in dom(f) \cap dom(g)$ and $f(x) \neq g(x)$ or (w.l.o.g.) $x \in dom(g) \setminus dom(f)$:

As $x \in dom(g)$, for any interval $O_{m,e}$ with $g(x) \in O_{m,e}$ there is an interval $C_{n,d}$ with $x \in C_{n,d} \subseteq dom(g)$ (open domain!) and $g(C_{n,d}) \subseteq O_{m,e}$ (continuous function!). So $FO_{n,d,m,e}$ is a neighborhood of g.

Let $f(x) \neq g(x)$. Then there are such intervals $O_{m,e}$ and $C_{n,d}$ with $f(x) \notin O_{m,e}$, so $f \notin FO_{n,d,m,e}$.

Let $x \notin dom(f)$. Then $f \notin FO_{n,d,m,e}$ for any $O_{m,e}$, $C_{n,d}$ with $x \in C_{n,d}$.

In any case there is (w.l.o.g.) a neighborhood g which is not a neighborhood of f. □

In order to be able to construct a σ with $\{\sigma(n) : n \in \mathbb{N}\} = \{FO_{n,d,m,e} : C_{n,d} \subseteq dom(f) \wedge fC_{n,d} \subseteq O_{m,e}\}$, it is sufficient to have a subset of the latter set, such that for any $x \in dom(f)$ and $m \in \mathbb{N}$ there is a pair $(C_{n,d}, O_{m,e})$ in this subset with $x \in O_{n,d}$. We will often implicitly use this property.

In order to generalize the framework for complex analysis, it is sufficient to use complex dyadic numbers $\{m \cdot 2^e + \mathbf{i} \cdot \overline{m} \cdot 2^{\overline{e}} \mid m, e, \overline{m}, \overline{e} \in \mathbb{Z}\}$ and corresponding disks $O_{n,d} = \{z \in \mathbb{C} : |z - d| < 2^{-n}\}$ and $C_{n,d} = \{z \in \mathbb{C} : |z - d| \leq 2^{-n}\}$ replacing the real dyadic numbers and intervals from above. $\varrho_{\mathbb{C}}$, $\tau_{\mathbb{C}}$, $\varrho_{S(\mathbb{C})}$, $\tau_{S(\mathbb{C})}$, $\varrho_{CO(\mathbb{C})}$, and $\tau_{CO(\mathbb{C})}$ will be the corresponding representations and topologies. When speaking of constructive or computable operators, this will always be with respect to these admissible representations.

Mechanisms to construct further admissible representations can be found e.g. in [We94]. We will (implicitly) use the following: If $\varrho_1, \ldots, \varrho_n$ are admissible representations of $M_1, \ldots, M_n$, then ϱ defined by $\varrho(\sigma) := (\varrho_1(\sigma_1), \ldots, \varrho_n(\sigma_n))$ is an admissible representation of $M_1 \times \ldots \times M_n$, where $\sigma_k(m) := \sigma(<k, m>)$. Here $<k, m>$ denotes the pairing function on $\mathbb{N}^2$.

3 Implementations

To yield wider acceptance of constructive analysis, especially in the field of numerical analysis, there should be adequate implementations in modern programming languages. Brent's MP-package [Br78] is too old, FORTRAN 66 is obsolete.

Although representations are a concept of high abstraction, they could easily be implemented as abstract data types. Oracle Turing machines on the other hand correspond to procedures with functional arguments, which are allowed in any modern imperative programming language, not to mention functional programming languages.

A few examples may suffice. Define types `NATURAL` for $\mathbb{N}$ and `DYADIC` for $\mathbb{D}$ using lists of integers and implement fast arithmetic on these sets, see e.g. [Sch94] for a Turing machine(!) implementation. On top of this basic arithmetic there should be a user interface which is simple and intuitive, but reflects the constructive approach:

Modern functional language like e.g. HASKELL allow declarations like the following:

```
type INTEGRAL          = [Int]
type DYADIC            = (INTEGRAL,INTEGRAL)
type REAL_NUMBER       = (NATURAL -> DYADIC)
type REAL_SEQUENCE     = (NATURAL -> REAL_NUMBER)
type REAL_FUNCTION     = (REAL_NUMBER -> REAL_NUMBER)
type REAL_OPERATOR     = (REAL_FUNCTION -> REAL_FUNCTION)
```

With object oriented programming and especially with overloading of operators allow expressions like `x**2 + sqrt(y)`, where `x` and `y` are of type `REAL_NUMBER` and `sqrt` is a `REAL_FUNCTION`!

In imperative languages like C or MODULA, it is impossible to compute values that are functions. So declarations like the following are necessary (and allowed):

```
REAL_NUMBER   = PROCEDURE (NATURAL):DYADIC
REAL_SEQUENCE = PROCEDURE (NATURAL, NATURAL):DYADIC
REAL_FUNCTION = PROCEDURE (REAL_NUMBER, NATURAL):DYADIC
REAL_OPERATOR = PROCEDURE (REAL_FUNCTION, REAL_NUMBER, NATURAL):DYADIC
```

The following chapters should also be read under this algorithmic viewpoint.

4 Examples of non-constructivity

Consider e.g. the set of converging power series

$$S_0 := \{(a_k) \in S(\mathbb{C}) \mid \text{ radius of convergence } > 0\}$$

and the set of functions analytic at least in 0:

$$A_0 := \{f \in CO(\mathbb{C}) \mid f \text{ analytic in } 0 \in dom(f)\}$$

4.1 *Theorem.* *There is no continuous operator $\Sigma_0 : S(\mathbb{C}) \dashrightarrow CO(\mathbb{C})$ summing up power series such that $\Sigma_0(S_0) \subseteq A_0$.* □

Proof: (by contradiction) Consider the sequence $(a_k) \equiv 0$ (so $(a_k) \in S_0$) and let $f := \Sigma_0((a_k)) \in A_0$. As $dom(f)$ is open and not empty there is a disk $C_{n,d} \subseteq dom(f)$. The pair $(C_{n,d}, O_{0,0})$ of disks defines a neighborhood $FO_{n,d,0,0}$ of f (in $\tau_{CO(\mathbb{C})}$). So there should be a neighborhood $\mathcal{U}$ of (a_k) such that $\Sigma_0(\mathcal{U}) \subseteq FO_{n,d,0,0}$. In consequence there must be a finite number of elements $s_1, s_2, \ldots, s_n$ of our subbase for $\tau_{S(\mathbb{C})}$ such that $s_1 \cap \ldots \cap s_n \subseteq \mathcal{U}$, so $\Sigma_0(s_1 \cap \ldots \cap s_n) \subseteq FO_{n,d,0,0}$. But this intersection obviously contains power series with nontrivial radius of convergence and arbitrarily large values on the disk $C_{n,d}$, e.g. power series for monomials $g(z) = c \cdot x^\ell$ for large ℓ and (larger) c.□

4.2 *Theorem.* *There is no continuous operator* $Seq_0 : CO(\mathbb{C}) \dashrightarrow S(\mathbb{C})$ *computing coefficients of power series in 0 such that* $Seq_0(A_0) \subseteq S_0$. □

Proof: (by contradiction) Consider $f \equiv 0$, $dom(f) = (-1, 1)$, and the sequence $(a_k) \equiv 0$. Choose the neighborhood $S_{2,1,0}$ of (a_k) (giving the information that a_2 should be in the disk $\{z : |z| < 1/2\}$). If Seq_0 were continuous, there should be a neighborhood $\mathcal{U}$ of f such that $Seq_0(\mathcal{U}) \subseteq S_{2,1,0}$. So there must be a corresponding finite set $(I_1, J_1), \ldots (I_n, J_n)$ such that $f(I_m) = \{0\} \subseteq J_m$ for any m. As the J_m are open, we are able to choose $\varepsilon > 0$ with $\{z : |z| \leq 2\varepsilon\} \subseteq J_m$. Now consider the function $\overline{f}$ defined by $dom(\overline{f}) = dom(f)$ and $\overline{f}(x) = \max\{0, \varepsilon - x^2\}$. So $\overline{f}$ is analytic on $\{z : |z| < \sqrt{\varepsilon}\}$ and $|\overline{f}(x)| \leq \varepsilon$ for any $z \in dom(f)$. Hence $\overline{f} \in \mathcal{U} \cap A_0$ and $\overline{f}'(0)/2! = 1$, so $Seq(\overline{f}) \notin S_{2,1,0}$. □

In consequence we will need representations giving additional informations in order to yield constructive solutions. There are two central properties that determine most aspects of constructivity, computability, and complexity concerning f and (a_k):

> (a) Consider a power series $\sum_{k=0}^{\infty} a_k \cdot z^k$ (with complex numbers a_k and z) converging absolutely for an arbitrary $z \neq 0$. As an immediate consequence there are $R_a \in \mathbb{R}^+$, $M_a \in \mathbb{R}^+$ such that $\sum |a_k| R_a^k \leq M_a$.

and

> (b) Consider f that is differentiable in a (complex) neighborhood of zero. As an immediate consequence there are $R_b \in \mathbb{R}^+$, $M_b \in \mathbb{R}^+$ such that f is differentiable on $\{z \in \mathbb{C} : |z| < R_b\}$ and $M_b \geq \max\{|f(z)| : |z| \leq R_b\}$.

There is a simple connection between (R_a, M_a) and (R_b, M_b), if (a_k) and f correspond: Given (R_a, M_a), we may choose $R_b := R_a$ and $M_b := M_a$. Given (R_b, M_b) and an arbitrary $\varepsilon \in (0, 1)$, we may choose $R_a := R_b \cdot (1 - \varepsilon)$ and $M_a = M_b/\varepsilon$.

Using the additional information given by (R_a, M_a) or (R_b, M_b) will be sufficient to find the constructive operators

$$((a_k), R_a, M_a) \ \mapsto \ (f, R_b, M_b)$$

and

$$(f, R_b, M_b) \ \mapsto \ ((a_k), R_a, M_a)$$

In consequence, many important operators on complex functions are constructive (and quite efficient!) when the functions are given as triples $((a_k), R_a, M_a)$ or (f, R_b, M_b). Among these are integration and solving initial value problems.

In the following let $PS(\mathbb{C}) \subseteq S(\mathbb{C}) \times \mathbb{R}^2$ be the set of all triples $((a_k), R, M)$ such that $\sum |a_k| R^k \leq M$ and let $AF(\mathbb{C}) \subseteq CO(\mathbb{C}) \times \mathbb{R}^2$ be the set of all triples (f, R, M) such that f is differentiable on $\{z \in \mathbb{C} : |z| < R\}$ and $M \geq \max\{|f(z)| : |z| \leq R\}$.

5 Summation of Power Series

Using the additional parameters R and M, it is quite easy to sum power series:

5.1 *Lemma.* *[Mu93] Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers, let $R, M \in \mathbb{R}$, $R, M > 0$, be given such that $\sum_{k\in\mathbb{N}} |a_k| R^k \le M$. For any $Q \in \mathbb{R}$, $Q > 1$, and for any $n \in \mathbb{N}$ let*

$$m(Q,n) := \lceil \frac{n + \log_2 M + 1}{\log_2 Q} \rceil.$$

Then for any $z \in \mathbb{C}$ with $|z| \le R/Q$

$$\sum_{k\in\mathbb{N}, k\ge m(Q,n)} a_k z^k \le 2^{-n-1}$$

□

Using this result, we are able to compute the sum function of the power series, given its coefficients and the parameters R and M, for any z with $|z| < R$.

In [Mu93] the complexity of the evaluation of the sum was examined depending on n, M, R, and Q. Nevertheless, M, R, and Q were not treated as 'inputs' of the algorithm, but rather like 'constant values', simplifying the analysis. In consequence, finite representations of these values were used: $M = 2^\sigma$, $R = 2^{-\tau}$, and $Q = 2^q$ for dyadic numbers σ, τ, and q.

Sometimes it would be close at hand to evaluate certain formulae with a precision of 2^{-n} for a *negative* n. Because our computational model is only defined for $n \ge 0$, the notation $[\![x]\!] := \min\{i \in \mathbb{N} \mid i \ge x\}$ was chosen.

5.2 *Theorem.* *[Mu93] Let $(a_k)_{k\in\mathbb{N}}$ be a sequence of complex numbers and let $f(z) := \sum a_k z^k$. Let a natural number σ and dyadic decimals τ, $q > 0$ with finite representations of length $\le \ell$ be given such that $\sum_{k\in\mathbb{N}} |a_k| 2^{-k\tau} \le 2^\sigma$. Then for any z such that $|z| \le 2^{-\tau-q}$, $f(z)$ can be approximated with an error not exceeding 2^{-n} using approximations to the first m coefficients a_k with a precision of $[\![4 + n + \mathrm{ld} m - k(\tau + q)]\!]$ each. Here $m := \lceil \frac{1}{q} \cdot (n + \sigma + 1) \rceil$. The computational complexity of approximating $f(z)$ is bounded by $\mathcal{O}(\, m \cdot \mathcal{M}(n + m + \sigma + m[\![\tau + q]\!]) + m \cdot (\mathrm{ld} n + \mathrm{ld}\sigma + \ell)\,)$* plus *the complexity of evaluating the coefficients a_k.* □

This result can immediately be reformulated in two ways: As a method to evaluate of power series and as a method to map a power series to its sum function.

5.3 *Theorem.* *There is a computable operator $\Sigma_1 : S(\mathbb{C}) \times \mathbb{R}^2 \times \mathbb{C} \dashrightarrow \mathbb{C}$ such that $\Sigma_1((a_k), R, M, z) = \sum a_k z^k$ for any $((a_k), R, M, z)$ with $((a_k), R, M) \in PS(\mathbb{C})$ and $|z| < R$.* □

5.4 *Theorem.* *There is a computable operator $\Sigma_2 : S(\mathbb{C}) \times \mathbb{R}^2 \dashrightarrow CO(\mathbb{C})$ summing up power series such that $PS(\mathbb{C}) \subseteq dom(\Sigma_2)$ and $f := \Sigma_2((a_k), R, M)$ has domain $\{z \in \mathbb{C} : |z| < R\}$ for any $((a_k), R, M) \in PS(\mathbb{C})$.* □

If $((a_k), R, M) \in PS(\mathbb{C})$, then $(f, R, M) \in AF(\mathbb{C})$ for $f := \Sigma_2((a_k), R, M)$, hence:

5.5 *Corollary.* *There is a computable operator $\Sigma_3 : S(\mathbb{C}) \times \mathbb{R}^2 \dashrightarrow CO(\mathbb{C}) \times \mathbb{R}^2$ summing up power series such that $PS(\mathbb{C}) \subseteq dom(\Sigma_3)$ and $\Sigma_3(PS(\mathbb{C})) \subseteq AF(\mathbb{C})$.* □

6 Computing Coefficients of Series

There are different methods for computing the Taylor coefficients of an analytic function: [KoFr88] uses the following method: $\sum a_k x^k = f(x)$ can be rewritten as $a_k = f(x)/x^k - \sum_{i<k} a_i x^{i-k} - \sum_{i>k} a_i x^{i-k}$. Using a small x this gives $a_k \approx f(x)/x^k - \sum_{i<k} a_i x^{i-k}$, so an iterative solution is possible.

In [Mu87], f was interpolated by suitable polynomials P such that $a_k = f^k(0)/k! \approx P^k(0)/k!$. The estimation of the interpolating error was based on the mid value theorem which is not applicable in complex analysis. So the result was only given for real functions f. A generalization to complex functions using Hermite's representation of the interpolating polynomial can be found in [Mu93]. Again we will cite the main results. Let $(f, R, M) \in AF(\mathbb{C})$ be given.

6.1 ***Lemma.*** *[Mu93] If P is the unique polynomial interpolating f at $m+1$ distinct points $z_0, \ldots, z_m$ with $|z_i| \le \varepsilon$, where $\varepsilon \in \mathbb{R}$, $0 < \varepsilon \le R/2$, then for any $k \le m$*

$$\left| a_k - \frac{1}{k!} \cdot P^{(k)}(0) \right| \le M \cdot \varepsilon^{m+1-k} \cdot \frac{4^{m+1}}{R^{m+1}}$$

□

In order to allow a fast evaluation of $P^{(k)}(0)$, we may use special points z_i. Using only *real* z_i implies that the result below holds for functions f that are given only on the real line. Furthermore we choose $m = 2k$. So we may use $z_i := (i-k) \cdot h$ for an arbitrary $h \in \mathbb{R}$, $h>0$, and any $i \in \mathbb{Z}$, $0 \le i \le 2k$. So the Lagrangian representation of interpolating polynomials could be used for the evaluation of the polynomial.

Similar to the summation of power series, finite representations of M and R were sufficient: $M = 2^\sigma$, $R = 2^{-\tau}$ even for natural σ and τ.

6.2 ***Theorem.*** *[Mu93]*
Let $f\colon \mathbb{C} \dashrightarrow \mathbb{C}$ be analytic in 0, let $2^{-\tau}$ be less than the radius of convergence of the corresponding power series and let $2^\sigma \ge \max\{|f(t)| : |t| = 2^{-\tau}\}$ for two natural numbers $\sigma, \tau > 0$. Then any coefficient a_k of the Taylor series can be approximated with an error not exceeding 2^{-n} by evaluating f at $2k+1$ points from the real interval $\{x \in \mathbb{R} : |x| \le 2^{-\tau}\}$ with a precision of $2n+2k\tau+13k+\sigma+6$ digits. The computational complexity of approximating a_k is bounded by $\mathcal{O}(\,(k+1) \cdot \mathcal{M}(n+k\tau+\sigma) + k^2 \cdot \mathcal{M}(k \mathrm{ld} k)\,)$ plus *the time necessary for the $2k+1$ evaluations of f.* □

A constructive version of this result can be formulated as follows:

6.3 ***Theorem.*** *There is a computable operator $Seq_1 : CO(\mathbb{C}) \times \mathbb{R}^2 \dashrightarrow S(\mathbb{C})$ computing the coefficients of power series such that $AF(\mathbb{C}) \subseteq dom(Seq_1)$.* □

If $(f, R, M) \in AF(\mathbb{C})$, then e.q. $((a_k), R/2, 2 \cdot M) \in AF(\mathbb{C})$ for $(a_k) := Seq_1(f, R, M)$, hence:

6.4 ***Corollary.*** *There is a computable operator $Seq_2 : CO(\mathbb{C}) \times \mathbb{R}^2 \dashrightarrow S(\mathbb{C}) \times \mathbb{R}^2$ computing the coefficients of power series such that $AF(\mathbb{C}) \subseteq dom(Seq_2)$ and $Seq_2(AF(\mathbb{C})) \subseteq PS(\mathbb{C})$.* □

If f is analytic on the whole of its domain, there even is a better result:

6.5 ***Theorem.*** *There is a computable operator $Seq_3 : CO(\mathbb{C}) \dashrightarrow S(\mathbb{C})$ computing the coefficients of power series such that $\{f \in CO(\mathbb{C}) : f$ analytic on $dom(f), 0 \in dom(f)\} \subseteq dom(Seq_3)$.* □

Proof: In our representation $\varrho_{CO(\mathbb{C})}$ a name for f consists of all pairs of open disks $(C_{n,d}, O_{m,e})$ with $C_{n,d} \subseteq dom(f)$ and $fC_{n,d} \subseteq O_{m,e}$. As $dom(f)$ is open, there must be such a pair with $0 \in C_{n,d}$, we may even assume $d = 0$. So we know that f is analytic on $\{z \in \mathbb{C} : |z| \leq 2^{-n}\}$ with $f(z) \leq |e| + 2^{-m}$, i.e. $(f, 2^{-n}, |e| + 2^{-m}) \in AF(\mathbb{C})$. □

Unfortunately, it is hard to generalize this result to an operator $Seq_3 : CO(\mathbb{C}) \dashrightarrow S(\mathbb{C}) \times \mathbb{R}^2$ with $Seq_3(f) = ((a_k), R, M) \in PS(\mathbb{C})$ for any analytic f: Given f, the sequence (a_k) is unique, but this is not true for R and M: Although the values 2^{-n}, $|e|+2^{-m}$ from above satisfy $(f, 2^{-n}, |e|+2^{-m}) \in AF(\mathbb{C})$, there are depending on the specific name for f. Of course, we might use this to define a kind of 'computable relation' (instead of a computable function).

7 Manipulating power series

As the previous chapters show, there are constructive and even quite efficient operators between analytic functions and their power series. So certain operators mapping functions to functions can be computed efficiently if we are able to do this evaluation by a transformation to power series, a manipulation of the series and a retransformation to a function. Among these are integration and solving initial value problems:

If $f(z) = \sum a_k z^k$, then $\int_0^z f(t)dt = \sum a'_k z^k$, where $a'_0 = 0$ and $a_{k+1} = a_k/(k+1)$. So a formal integration is easy. In addition, if $\sum |a_k R^k| \leq M$, then $\sum |a'_k R^k| = \sum |a_k R^{k+1}/(k+1)| \leq \sum |a_k R^{k+1}| \leq R \cdot M$.

Therefore, the following way of integration is constructive. (Of course, $R/2$ and $2M$ may be replaced by $R \cdot (1 - \varepsilon)$ and M/ε, see chapter 4.)

$$\begin{aligned} (f, R, M) \in AF(\mathbb{C}) &\longrightarrow ((a_k), R/2, 2M) \in PS(\mathbb{C}) \\ &\longrightarrow ((a'_k), R/2, R \cdot 2M) \in PS(\mathbb{C}) \\ &\longrightarrow (\textstyle\int f, R/2, R \cdot 2M) \in AF(\mathbb{C}) \end{aligned}$$

As shown in [Mu87], this essentially leads from complexity $t(n)$ for f to $n^2 \cdot t(n)$ for $\int f$, if t satisfies certain (weak) conditions. Compare this to the exponential complexity of integration that important numerical algorithms imply!

Initial value problems can be solved in a similar way: Suppose $f(z, w) = \sum c_{i,j} z^i w^j$ and consider $w = w(z)$ such that $f(z, w(z)) = w'(z)$ and $w(0) = 0$. Then $w(z) = \sum a_k z^k$ is analytic and comparing coefficients in the identity $\sum c_{i,j} z^i (\sum a_k z^k)^j = \sum a_k z^{k-1}$ leads to a recursive, polynomial-time algorithm for (a_k) [MuMo93].

Here we have to deal with functions $f : \mathbb{C}^2 \to \mathbb{C}$ and two-dimensional series, where appropriate representations $\varrho_{CO(\mathbb{C}^2)}$ of $CO(\mathbb{C}^2)$ and $\varrho_{S(\mathbb{C}^2)}$ of $S(\mathbb{C}^2)$ can easily be defined. As an analogon to $PS(\mathbb{C})$ and $AF(\mathbb{C})$ we have $PS(\mathbb{C}^2) = \{((a_{i,j}), R_1, R_2, M) \mid \sum |a_{i,j} \cdot R_1^i \cdot R_2^j| \leq M\}$ and $AF(\mathbb{C}^2) = \{((f, R_1, R_2, M)|f$ is differentiable on $\{(z, w) : |z| < R_1, |w| < R_2\} \wedge M \geq \max\{|f(z)| : |z| \leq R_1, |w| \leq R_2\}\ \}$

So we have the following way for efficient solutions of initial value problems (for the details of the construction see [MuMo93]) :

$$\begin{aligned} (f, R_1, R_2, M) \in AF(\mathbb{C}^2) &\longrightarrow ((a_{i,j}), R_1/2, R_2/2, 4 \cdot M) \in PS(\mathbb{C}^2) \\ &\longrightarrow ((c_k), R_3, R_3 \cdot 4 \cdot M) \in PS(\mathbb{C}) \\ &\longrightarrow (w, R_3, R_3 \cdot 4 \cdot M) \in AF(\mathbb{C}) \end{aligned}$$

where $R_3 := \min\{R_1/2, R_2/(8M)\}$. ($R_i/2$ and $4M$ may be replaced by $R_i \cdot (1 - \varepsilon)$ and M/ε^2.)

8 Analytic continuation

The result on the summation of power series showed, that given $((a_k), R, M)$ we may compute the sum function f on $\{z : |z| < R\}$. But if f if defined and analytic on a open and connected set, then the power series of f in a single point of the domain uniquely determines f on the whole of $dom(f)$, even it there is no single circle of convergence covering this domain. So the question is: What is the necessary information to compute f on its domain? First, there is a negative answer. The 'local' information given by $((a_k), R, M)$ is not sufficient!

8.1 *Theorem.* *There is no continuous operator $\Sigma_4 : S(\mathbb{C}) \times \mathbb{R}^2 \dashrightarrow CO(\mathbb{C})$ summing up power series such that $PS(\mathbb{C}) \subseteq dom(\Sigma_4)$ and such that $\{z \in \mathbb{C} : |z| < R\}$ is a proper subset of $dom(f)$ for some $((a_k), R, M) \in PS(\mathbb{C})$, where $f := \Sigma_4((a_k), R, M)$.* □

Proof: (by contradiction) Suppose Σ_4 is continuous and let $((a_k), R, M) \in PS(\mathbb{C})$ be given such that $\{z \in \mathbb{C} : |z| < R\}$ is a proper subset of $dom(f)$ for $f := \Sigma_4((a_k), R, M)$.

As $dom(f)$ is open, the (open!) set $dom(f) \cap \{z \in \mathbb{C} : |z| > R\}$ must be nonempty. So there is a disk $O_{i,d} \subseteq dom(f)$ with $O_{i,d} \cap \{z \in \mathbb{C} : |z| < R\} = \emptyset$. Here $|d| > R$ must hold!

f is continuous, so for d there must exist $C_{n,d}$ and $O_{m,e}$ with $C_{n,d} \subseteq O_{i,d} \subseteq dom(f)$ and $fC_{n,d} \subseteq O_{m,e}$. Hence $f \in FO_{n,d,m,e}$.

Σ_4 is continuous, so there are neighborhoods $\mathcal{U}_S$ of (a_k), $\mathcal{U}_R$ of R, and $\mathcal{U}_M$ of M, such that $\Sigma_4(\mathcal{U}_S \times \mathcal{U}_R \times \mathcal{U}_M) \subseteq FO_{n,d,m,e}$. In addition, there must be an $\varepsilon > 0$ with $M + \varepsilon \in \mathcal{U}_M$, a $\delta > 0$ with $R - \delta \in \mathcal{U}_R$ and an $\ell \in \mathbb{N}$ such that $(a'_k) \in \mathcal{U}_S$ for any sequence (a'_k) satisfying $a_k = a'_k$ for $k < \ell$.

For arbitrary $j \geq \ell$ consider the function $g_j(z) := (z/R)^j$ and the sequence $(a_k^{(j)})$ with $a_j^{(j)} = a_j + R^{-j}$ and $a_k^{(j)} = a_k$ if $k \neq j$. Obviously, g_j is analytic (even on $\mathbb{C}$). In addition, $\lim_{j\to\infty} |g_j(d)| = \infty$ and $\lim_{j\to\infty} g_j(R-\delta) = 0$. So there is a j with both $\sum |a_k^{(j)}(R-\delta)^k| \leq |g_j(R-\delta)| + \sum |a_k R^k| \leq \varepsilon + M$ and $f(d) + g_j(d) \notin O_{m,e}$.

So $((a_k^{(j)}), R - \delta, M + \varepsilon) \in PS(\mathbb{C}) \cap (\mathcal{U}_S \times \mathcal{U}_R \times \mathcal{U}_M)$, but $\Sigma_4((a_k^{(j)}), R - \delta, M + \varepsilon) \notin FO_{n,d,m,e}$. □

So for a constructive analytic continuation, we must have a kind of global information on (R_x, M_x) for any $x \in dom(f)$, e.g. in the following form:

8.2 *Theorem.* *There is a computable operator $\Sigma_5 : S(\mathbb{C}) \times CO(\mathbb{C}) \dashrightarrow CO(\mathbb{C})$ with following property: If $((a_k), \overline{M}) \in S(\mathbb{C}) \times CO(\mathbb{C})$ satisfies*

(a) $dom(\overline{M})$ is connected,
(b) there is an $f \in CO(\mathbb{C})$ with domain $dom(f) = dom(\overline{M})$,
(c) f is analytic on $dom(\overline{f})$,
(d) $|f(z')| \leq \overline{M}(z) \in \mathbb{R}$ for any $z \in dom(\overline{M})$, and
(e) (a_k) is the Taylor series of f in 0,

then $\Sigma_5((a_k), \overline{M}) = f$. □

Proof: From any name for $\overline{M}$ in $\varrho_{CO(\mathbb{C})}$, we may derive a sequence S_1 of pairs n_i, d_i such that $C_{n_i,d_i} \subseteq dom(\overline{M})$ and $\bigcup_{i\in\mathbb{N}} O_{n_i,d_i} = dom(\overline{M})$. In addition, we get a sequence M_1 of dyadic numbers m_i with $\overline{M}(z) \leq m_i$ for any $z \in O_{n_i,d_i}$.

Due to property (a), we may use sequence S_1 to construct a sequence S_2 of pairs (n'_i, d'_i) with $C_{n'_i,d'_i} \subseteq dom(\overline{M})$ and $\bigcup_{i\in\mathbb{N}} O_{n'_i,d'_i} = dom(\overline{M})$ such that additionally $d'_0 = 0$, and especially $d'_i \in \bigcup_{j<i} O_{n'_j,d'_j}$ for $i > 0$ hold: Enumerate $\mathbb{N}^2 \times \mathbb{S}$ and, for any (k, n, d), test whether $C_{n,d} \subseteq (\bigcup_{i\leq k} O_{n_i,d_i})$. If this is true and $d = 0$ or d is in the union of open disks already constructed for S_2, then add (n, d) to S_2. Proving $\bigcup_{i\in\mathbb{N}} O_{n'_i,d'_i} = dom(\overline{M})$ can be done with standard arguments for covering sets. Together with S_2, we may construct a sequence M_2 of dyadic numbers m'_i with $\overline{M}(z) \leq m'_i$ for any $z \in C_{n'_i,d'_i}$.

Due to properties (b) and (c), f is analytic on these disks $O_{n'_i,d'_i}$. With (d) we know that even $|f(z)| \leq m'_i$ for any $z \in C_{n'_i,d'_i}$. So now we have the 'local' information!

Using property (e), (a_k), n'_0, and m'_0, we may compute f on $O_{n'_0,d'_0}$. As $d_1 \in O_{n'_0,d'_0}$, we are able to use n'_0 and m'_0 to compute $(a_k^{(1)})$ such that $\sum a_k^{(1)}(z - d_1)^k = f(z)$ holds in some neighborhood of d'_1. Using n'_1 and m'_1, we may even compute $f(z)$ on $O_{n'_1,d'_1}$. Using this argument in an inductive way shows computability of f on any $O_{n'_i,d'_i}$, so f is computable on $dom(\overline{M}) = dom(f)$. □

So an analytic function f is determined in a constructive way by a single power series and a function dominating f. Please note that for our representation this dominating function has information on the domain! A reverse transformation $f \mapsto ((a_k), \overline{M})$ holds with Theorem 6.5 and $\overline{M}(z) := |f(z)|$.

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